VISCOUS INSTABILITY OF AN INCOMPRESSIBLE PLANE JET

V. N. Kalugin and G. A. Chechko

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The stability problem is solved numerically for an incompressible plane jet. The minimum critical Reynolds number is determined. A comparison is drawn with result obtained by asymptotic methods.

Many authors have pursued theoretical and experimental investigations of the stability of incompressible jets. Experiments have shown that loss of stability is observed for Reynolds numbers from 10 to 30 [1]. Theoretical analyses based on asymptotic methods, on the other hand, have led to extremely disparate results, even indicating nonexistence of the lower branch of the neutral curve. This inconsistency is clearly attributable to the diversity of assumptions on which the approximate analytical calculations are based. We now carry out a numerical analysis of the complete Orr–Sommerfeld equation, using the method of [2].

We consider the stability of a two-dimensional laminar jet issuing from a narrow slot into a space filled with a fluid at rest. The longitudinal velocity component of the jet has the form [1]

$$u = U_0^* \operatorname{sech}^2 ay, \tag{1}$$

where $U_0^* = (3M^2/32\rho_*\nu_*x_*)^{1/3}$ is the characteristic flow velocity, $L_* = a\,(M/48\rho_*\nu_*^2x_*^2)$ is the characteristic length, $M = \int\limits_{-\infty}^{+\infty} \rho_* u_*^2 dy_* = \text{const}$ is the momentum flux of the jet in the direction of the x_* axis, ρ_* is the density of the fluid, ν_* is the kinematic viscosity coefficient, a is a parameter characterizing the width of the domain of integration, $y = y_*/L_*$, and $x = x_*/L_*$.

The Orr–Sommerfeld equation with respect to the complex amplitude of the perturbed-flow stream function is written as follows in dimensionless variables:

$$(U-c)(\varphi'' - \alpha^2\varphi) - U''\varphi = -\frac{i}{\alpha R}(\varphi^{IV} - 2\alpha^2\varphi^{II} + \alpha^4\varphi), \tag{2}$$

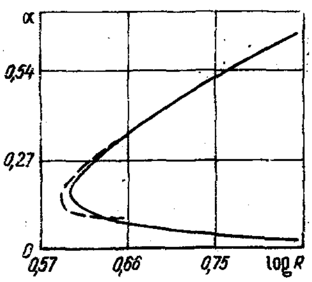

Fig. 1

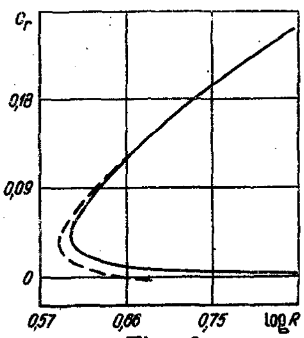

Fig. 2

Fig. 1. Neutral curves for $a = 4$ (dashed) and $a = 8$ (solid).

Fig. 2. Values of c_r on neutral curve for $a = 4$ (dashed) and $a = 8$ (solid).

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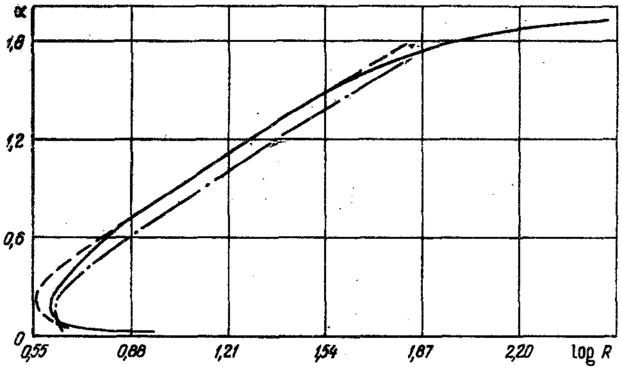

Fig. 3. Neutral curves according to: authors (solid); Tatsumi and Kakutani (dashed); Clenshaw and Elliott (dot-dash).

where $R = U_0^* L_* / \nu_*$ is the Reynolds number, $u(y) = u_* U_0^*$ is the dimensionless velocity profile, $c = c_r + ic_i$ is the complex phase velocity, c_r is the perturbation wave propagation velocity, and c_i is the growth (decay) rate.

Inasmuch as the investigated flow is symmetrical about the y axis, the even and odd solutions $\varphi(y)$ can be sought independently. We know from [3] that antisymmetrical perturbations (φ even) are considered to be the most dangerous, and we therefore adopt as the boundary conditions on the symmetry axis y = 0

$$\varphi'(0) = \varphi'''(0) = 0. \tag{3}$$

The boundary conditions for Eq. (2) when $y \to \infty$ are

$$\varphi(\infty) = \varphi'(\infty) = 0. \tag{4}$$

The numerical calculations are carried out in the finite domain $0 \le y \le 1$. Here the parameter a is made large enough to permit the assumption U = 0 for $y \ge 1$. Then the complete solution of Eq. (2) in the domain $y \ge 1$ with allowance for the boundedness condition at infinity has the form

$$\varphi = C_1 e^{-\beta y} + C_2 e^{-\alpha y}, \tag{5}$$

where C_1 and C_2 are arbitrary constants, $\beta = \sqrt{\alpha^2 - i\alpha Rc}$, and $\beta_r > 0$.

Following [2], we replace Eq. (2) by a system of nonlinear differential equations in the functions q, f, Ψ, and Φ, the numerical integration of this system is devoid of the problems that arise in the numerical solution of Eq. (2). The condition that the known outer solution (5) and $y \ge 1$ merge with the solution of (2) inside the domain $0 \le y \le 1$ implies the following conditions at y = 1:

$$q(1) = \beta,\ f(1) = \beta^2 - \alpha^2,\ f'(1) = 0,\ \Psi(1) = \frac{1}{\alpha + \beta},\quad \Phi(1) = \alpha. \tag{6}$$

If conditions (3) are to be satisfied on the axis, the new functions q, f, Ψ, and Φ must, as inferred from [4], satisfy the relation

$$\Phi q - f'\Psi = 0, \tag{7}$$

where q, f, Ψ, and Φ are evaluated at the point y = 0.

We have thus reduced the problem to finding values of the parameters α, R, and c such that the left-hand side of Eq. (7) will be equal to zero. To do so in the present article we use the procedure and iterative schemes proposed in [2, 4, 5]. The system of equations for the functions q, f, Ψ, and Φ is integrated from the outer edge of the domain y = 1.

The neutral curves calculated for two different values of the parameter a = 4 and 8 practically coincide and are highly consistent with the analytical results of Tatsumi and Kakutani [6] over a wide range of large wave numbers (Figs. 1 and 3). It is evident from Fig. 2 that as α tends to zero the neutral curve obtained for a = 4 enters into the domain of negative values of c_r, but this excursion is unallowable from physical considerations and is caused by the fact that the domain of integration was not made wide enough. Thus, for small values of α the value of c_r becomes small, and the critical point $u = c_r$ is shifted upward along the y axis, "leaving" the domain of integration. With a decrease of α, therefore, it is necessary to increase the width of the domain of integration (see Fig. 2). Now the eigenvalues for large values of α remain practically invariant as a is increased. Thus, the minimum critical Reynolds for $a > 8$ does not differ from that obtained for $a = 8$ to the fourth significant figure. We have for the minimum Reynolds number normalized to a domain width $a = 1$

$$R_{\min} = (4.5Mx_* \nu_*^{-2})^{\frac{1}{3}} = 3.981.$$

Consequently, the minumum distance to which the jet remains laminar is determined from the expression

$$x_* = \frac{(3.981)^3 \nu_*^2}{4.5M}.$$

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